

The McMullen Map In Positive Characteristic

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1 Introduction

A rational function over a field K can be studied by looking at its fixed and periodic points. At a finite fixed point z of a rational function φ , the derivative $\varphi'(z)$ can be interpreted as the map on tangent spaces, and is conjugation-invariant: that is, if A is any linear-fractional transformation, then $\varphi'(z) = (A\varphi A^{-1})'(z)$. We call this conjugation-invariant derivative the **multiplier** of φ at z . If z is the point at infinity, we can extend the definition of the multiplier by conjugating it to be any finite point. We can furthermore define the multiplier of a point of exact period n to be $(\varphi^n)'(z)$; here and in the sequel, powers of maps denote iteration rather than multiplication. The set of multipliers of period n , called the **multiplier spectrum**, depends only on the map φ , and because it is conjugation-invariant it really only depends on its conjugacy class.

Thus the symmetric functions in the multipliers are regular algebraic functions on the space of rational maps of degree d modulo conjugation, M_d . Collecting the symmetric functions in the period- n multipliers for each n , we obtain regular algebraic maps Λ_n from M_d to large affine spaces \mathbb{A}^{K_n} where K_n is the number of period- n points. Because there are infinitely such maps, one for each n , a priori we would expect there to be some large n for which the map

$$\Lambda_1 \times \dots \times \Lambda_n : M_d \rightarrow \mathbb{A}^{K_1 + \dots + K_n}$$

is injective. This would allow us to analyze the geometry of M_d purely in terms of multipliers. The map is not injective, but we do have a partial result:

Theorem 1.1. *Over a field of characteristic 0 or greater than d , for sufficiently large n the map*

$$\Lambda_1 \times \dots \times \Lambda_n : M_d \rightarrow \mathbb{A}^{K_1 + \dots + K_n}$$

is finite-to-one outside a proper closed subset of M_d .

Unfortunately, we only get generic finiteness. The following example provides a family of functions in M_d with the same multipliers, which we call an **isospectral family**:

Definition 1.2. The map $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is called a **Lattès map** if there is an elliptic curve E , a finite morphism $\alpha : E \rightarrow E$, and a finite separable map π such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & E \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{P}^1 & \xrightarrow{\varphi} & \mathbb{P}^1 \end{array}$$

If we choose π to be the projection by $P \sim -P$, then α must be of the form $\alpha_{m,T} : P \mapsto mP + T$ where $T \in E[2]$.

McMullen has proven the following result:

Theorem 1.3. [11] *Excluding the Lattès maps, there are no positive-dimension isospectral families over \mathbb{C} .*

McMullen's result is stronger than Theorem 1.1 in characteristic zero. However, his proof uses inherently complex-analytic methods. Although it is true over any characteristic-0 field by the Lefschetz principle, it does not give us any proof in positive characteristic. In fact, the result is false in characteristic p , as shown in,

Example 1.4. Let $\varphi(z) = \psi(z^p) + az$ where a is a constant and ψ is a family of rational functions. The family is isospectral (all period- n multipliers at finite points are a^n). However, if ψ is large enough, for example if $\dim \psi > 3$, then the family has no hope of reducing to a point in M_d since $M_d = \text{Rat}_d / \text{PGL}_2$ where Rat_d is the space of rational functions of degree d and PGL_2 acts by coordinate change.

However, Example 1.4 involves wild ramification. There are theoretical reasons to believe that a positive-characteristic version of Theorem 1.3 is true for tamely ramified maps. Briefly, McMullen's theorem is intimately connected with Thurston's rigidity theorem. Specifically:

Definition 1.5. A point z is a **preperiodic point** of φ if for some n , $\varphi^n(z)$ is a periodic point; we say the minimal such n is the **tail length**. Observe that z is preperiodic if and only if it has finite forward orbit. If all of the critical points of φ are preperiodic, we say that φ is **postcritically finite**, or in short PCF.

There are $2d - 2$ critical points counted with multiplicity when φ is tamely ramified, and so the condition that φ is PCF, at least if we fix the cycle and tail length of each critical point, is a set of $2d - 2$ algebraic equations. Since $\dim M_d = 2d - 2$, there should be finitely many PCF maps for each degree, cycle length, and tail length. Over \mathbb{C} , Thurston has proven that this is more or less the case:

Theorem 1.6. [6, 8] (*Thurston's Rigidity*) *Excluding the Lattès maps, all of which are PCF, there are no positive-dimension PCF families. Furthermore, the PCF equations meet transversally, i.e. fixing a cycle and tail length, the scheme of non-Lattès PCF maps is a finite set of reduced points.*

It is not a coincidence that the Lattès maps form the sole counterexample to both Thurston's rigidity and McMullen's theorem. McMullen's proof heavily uses rigidity. Briefly, his method in [11] is to assume an isospectral family exists, and then label an infinite set of so-called **repelling points**, that is periodic points whose multiplier λ satisfies $|\lambda| > 1$, accumulating at a point in the Julia set. A crucial fact about isospectral families is that their periodic points move without collision with one another or with critical points. Using previous results showing that this infinite set moves holomorphically, he shows that a critical point cannot pass through a point that is preperiodic to a repelling point; since there are infinitely many repelling points over \mathbb{C} , this implies that in an isospectral family all critical points are preperiodic, and then Thurston's rigidity shows that such a family is Lattès or trivial.

We will follow a similar program. Many of the methods of complex analysis have been successfully ported to the non-archimedean setting, and although a full version of Thurston's rigidity is beyond our reach, a sufficiently good approximation is proven in [1]. First, we have,

Definition 1.7. Let φ be defined over a complete algebraically closed non-archimedean field K . We say that φ is **tame** if the local degree at any analytic disk is not divisible by the residue characteristic p of K . Equivalently, $\varphi = f/g$ is tame if and only if for each integral model of φ over \mathcal{O}_K , the reduction mod the maximal ideal is tamely ramified after clearing common factors of the reductions of f and g . If φ is defined over a global field, we say it is tame if it is tame when we regard it as a map over every completion.

Remark 1.8. Whenever $\deg \varphi < p$, or whenever φ is a composition of maps of degree less than p , φ is tame.

We have as a prior result [1],

Theorem 1.9. *Let $\varphi(z) \in K(z)$ where K is any field such that $\text{char } K = p$. If φ is PCF and tame at every place of K , then the multipliers of φ are all defined over $\overline{\mathbb{F}}_p$.*

Remark 1.10. An immediate corollary of Theorem 1.9 is that if we can prove McMullen's theorem for a family, then we can prove the finiteness portion of Thurston's rigidity for it as long as the local degrees are not divisible by p . In [12] it is proven that Λ_1 is an isomorphism over \mathbb{Z} from M_2 to a plane in \mathbb{A}^3 , and thus Thurston's rigidity is also true for quadratic maps in characteristic 0 or $p \geq 3$.

Like McMullen, we will prove that in an isospectral family we can label a set of periodic points that accumulate at a point that is preperiodic to a repelling point, and that if a critical point passes through the preperiodic point it is equal

to it for all maps in the family. This will suffice to show that an isospectral map is PCF. This requires a repelling point in \mathbb{P}_K^1 where K is a complete algebraically closed characteristic- p field; unlike in the complex case, a repelling point is not guaranteed to exist. However, if we start from a global field, if the multipliers are not all in $\overline{\mathbb{F}_p}$ then we can complete with respect to a valuation that occurs in the denominator of a multiplier.

Since we do analysis, we in fact cannot start with a family defined over \mathbb{F}_p or any of its algebraic extensions. The best we can do is fix our field K to be a completion of $\overline{\mathbb{F}_p}(t)$, and then assume a family over K is isospectral. In other words, it can be viewed as an isospectral family of families. We will prove,

Theorem 1.11. *Suppose that C is an isospectral, tame family over K with the property that the multipliers are not all contained in $\overline{\mathbb{F}_p}$. Then C is trivial.*

Remark 1.12. The Lattès family has multipliers defined over $\overline{\mathbb{F}_p}$, and so is automatically excluded from the hypotheses of the theorem.

Now, consider a nontrivial family of spectra in $\mathbb{A}^{K_1+\dots+K_n}(\overline{\mathbb{F}_p})$. Theorem 1.11 tells us immediately that it cannot have a higher-dimension preimage in M_d . In particular,

Corollary 1.13. *The locus of spectra for which there is a positive-dimension isospectral family of tamely ramified maps is a finite set of points in $\mathbb{A}^{K_1+\dots+K_n}$, including the Lattès multipliers.*

And since it is easy to exhibit maps with distinct multipliers—for example, $z^d + 1$ has just one fixed point that is also critical whereas z^d has two—this gives us Theorem 1.1.

The result is not constructive: it does not tell us what n we need to choose to ensure that the map $\Lambda_1 \times \dots \times \Lambda_n$ will be finite-to-one, nor does it tell us the degree of such a map. Unlike in McMullen’s characteristic 0 case, Theorem 1.1 does not even tell us what the exceptions are to finiteness, i.e. what the finitely many spectra with positive-dimension preimages in Corollary 1.13 are. For some partial results in that direction, see [10, 12].

Effective versions of Theorem 1.1 would require different techniques. Unlike the techniques in [10], we are unable to list the finite set of exceptions in Corollary 1.13 (or any proper subset of M_d that contains all of them), although we conjecture that the Lattès family is the only one. Nor can we use these techniques to find or even bound the degree of the McMullen map, or the minimal period n such that $\Lambda_1 \times \dots \times \Lambda_n$ is generically finite.

We spend Sections 2 and 3 proving Theorem 1.11. In Section 2 we assume the existence of an infinite labeled set of repelling periodic points with the required property of moving without collision, and use that to prove that an isospectral family is PCF. This proof is easier than in the complex case, since non-archimedean analysis is more rigid than complex analysis, making it easier to show that a critical point cannot move very freely in an isospectral family.

In Section 3 we prove a key lemma that is left unproven in Section 2, allowing us to mark an infinite set of repelling points with just a finite amount of data.

The main result of Section 3 is a non-archimedean analog of constructing a local inverse of the projection to a Riemann surface from its universal cover; since a topological universal cover does not exist in non-archimedean analysis, we show that we can instead find a single neighborhood of an algebraic curve with local inverse maps to any unramified cover, as long as the curve is an isospectral family of tame maps.

2 Proof Outline

Let C be an isospectral irreducible curve as in the hypotheses of Theorem 1.11. Let K be a complete algebraically closed field such that one of the multipliers of the family C is repelling. By a result of Bézivin [5], this means that there are infinitely many repelling points. Moreover, the Julia set of φ is the closure of the set of repelling points. Bézivin's paper assumes characteristic 0, but the proofs and the result port perfectly to positive characteristic as long as the map is tamely ramified.

Moreover, repelling points move without collision: that is, if we mark any number of them, they will stay distinct, since a fixed point is repeated iff it has multiplier 1. McMullen's proof in fact hinges on using this to show that the repelling points, and with them the entire Julia set, move rigidly, so that after a local analytic conjugation the Julia set is constant in the family.

We now mark all critical points and enough other points to obtain an enhanced moduli space $M_d(\Gamma)$ so that there are no nontrivial automorphisms of any map in the family $C' = C \times_{M_d} M_d(\Gamma)$. The natural projection $M_d(\Gamma) \rightarrow M_d$ is finite to one, and by abuse of notation we denote the multiplier map on $M_d(\Gamma)$ by Λ_n , sending each enhanced φ to an ordered set of the multipliers of all the marked period- n points and the spectrum of the multipliers of the unmarked points. For example, if we mark all critical points and values, then there are automorphisms if and only if there are exactly 2 distinct ones, which is the case only if φ is conjugate to $z^{\pm d}$, in which case the multipliers lie in \mathbb{F}_p and the map is excluded from the hypotheses of Theorem 1.11.

The upshot is that,

Lemma 2.1. *Suppose that C' is chosen in such a way that for all $\varphi \in C'$, $\text{Aut } \varphi$ is trivial. Let $C'' = C' \times_{M_d(\Gamma)} M_d(\Gamma')$ where Γ' enhances Γ with one additional repelling periodic point. Then the natural projection $\pi : C'' \rightarrow C'$ is unramified.*

Proof. First, let n be the period, and let f be the polynomial whose roots are the period- n points that are unmarked in Γ . The repelling points are simple roots of f since repeated roots have multiplier 1, so ramification can only occur if there exists some automorphism of φ mapping z_1 to z_2 where z_1 and z_2 are both roots of f . Since φ has no automorphisms preserving the enhanced data of Γ , there is no ramification. \square

Now, let us change coordinates so that 0 is persistently preperiodic to a repelling point. After iteration, we may in fact assume that $\varphi(0)$ is fixed, and

for simplicity we will conjugate so that $\varphi(0) = \infty$. Since 0 is in the Julia set, there is a sequence of repelling periodic points converging to it for every map in the family.

Recall now the following results from non-archimedean analysis:

Proposition 2.2. *Let C be a curve. Let $x \in C$, and let $D \ni x$ be a neighborhood of x in C that is analytically isomorphic to a disk in \mathbb{P}^1 . The following functions are analytic on D :*

1. *The restriction to D of any rational map $f : C \rightarrow X$ for any curve X .*
2. *Any g from D to C' where $f : C' \rightarrow C$ is a rational map and $f \circ g$ is the identity on D and f maps $g(D)$ to D with local degree 1.*

Observe also that if in a projection map from C' to C a point $x \in C$ is unramified, there exists a neighborhood of x in C with a preimage disk that maps onto it with degree 1, so that we can apply the second case of Proposition 2.2 and obtain a local inverse. We have,

Lemma 2.3. *Let C' and C'' be as in the hypotheses of Lemma 2.1, and let $x \in C'$. There exists a disk $D \ni x$ in C' that is analytically isomorphic to a \mathbb{P}^1 -disk, depending only on C' , with the following property: the preimage of D in C'' is a disjoint union of \mathbb{P}^1 -disks each mapping into D with degree 1.*

We will delay the proof of Lemma 2.3 until Section 3. For now, observe that as a trivial consequence of joining Lemma 2.3 and Proposition 2.2, we obtain,

Corollary 2.4. *Suppose that 0 is a persistently Julia point. For each point $\varphi \in C'$, there exists an infinite set of repelling points z_i that converge to 0, such that for some neighborhood of $D \ni \varphi$, the functions z_i are all analytic on D . Moreover, the convergence of z_i is uniform on D .*

Proof. By repelling density, there is a sequence of repelling points z_i converging to 0. By Lemma 2.3, the hypotheses of Proposition 2.2 are satisfied and each repelling point is an analytic map from D to a neighborhood D' in C'' . Now the map θ from C'' to \mathbb{P}^1 defined by z_i is rational, and again using Proposition 2.2 it is analytic. Now, regardless of whether 0 is periodic, it cannot collide with any of the points z_i for any map in D . Thus, $z_i(D) \neq 0$ and so since z_i is analytic it maps D into a disk $D(a_i, r_i)$ where $r_i < |a_i|$ and since the absolute value is constant on each such disk we have $a_i \rightarrow 0$ and thus $z_i \rightarrow 0$ uniformly on D . \square

An infinite labeled sequence of analytic functions converging to 0 is the best we can hope for. We use the non-collision of periodic points to further prove

Lemma 2.5. *Suppose that a critical point x is preperiodic to a repelling cycle for some point on C' . Then it is preperiodic, with the same cycle and tail lengths, for all points on C' .*

Proof. After iteration, we may assume that $\varphi(x)$ is a fixed point. We will show that this remains true under infinitesimal perturbation; since the property of mapping to a fixed point is algebraic, cut out by the single equation $\varphi^2(x) = \varphi(x)$, it holds either for finitely many points on C' or on all of C' .

We assume that ∞ is repelling and that 0 maps to ∞ for all $\varphi \in C'$. Because we marked the critical points on C' , the function x is rational on C' , and its image misses ∞ , and so it maps each disk in C' to a disk analytically. Suppose that for some $\varphi_0 \in C'$, $x = 0$, and take a neighborhood of $D \ni \varphi_0$ that is isomorphic to a \mathbb{P}^1 -disk, such that x becomes an analytic function on D . Let us conjugate D to $D(0, 1)$ and φ_0 to $0 \in D(0, 1)$, so that $x(0) = 0$, and denote the map corresponding to $c \in D(0, 1)$ by φ_c .

Now, by Corollary 2.4, there exists an infinite sequence of analytic functions z_i on D such that $z_i(c)$ is a repelling periodic point for φ_c and $z_i(c) \rightarrow 0$ uniformly. From the proof of Corollary 2.4, $|z_i(c)| = |z_i(0)|$ for all i and $c \in D(0, 1)$.

Moreover, $z_i(c) - x(c)$ is an analytic function that is never zero: the multiplier at z_i is not zero, which makes it impossible for it to collide with a critical point. Now $z_i(c) - x(c)$ maps over a disk that includes $z_i(0)$ and excludes 0, so that $|z_i(c) - x(c)| \leq |z_i(0)|$.

Finally, we obtain

$$|x(c)| \leq \max\{|z_i(0)|, |z_i(c)|\} = |z_i(0)| \rightarrow_{i \rightarrow \infty} 0$$

and by the squeeze theorem, $x(c) = 0$ on D , hence on all of C' . \square

The next step is almost identical to the complex case:

Lemma 2.6. *All critical points of all maps in C are persistently preperiodic, with uniformly bounded cycle and tail lengths.*

Proof. We consider each critical point separately. If it is periodic, then it is persistently so because C' is isospectral, and the period is constant. If it is preperiodic to a repelling cycle, then the same is also true by Lemma 2.5. Let us now assume it is not preperiodic to a repelling cycle and derive a contradiction. There is a rational function x from C' to \mathbb{P}^1 sending a map to a critical point, and then the functions $\varphi^i(x)$ are rational for all i . Those functions all miss the repelling points, so if we fix any three repelling points, they become functions from C' to \mathbb{P}^1 minus three points. Recall that,

Lemma 2.7. *There are finitely many nonconstant separable maps from an affine algebraic curve to \mathbb{P}^1 minus three points.*

Proof. Write the algebraic curve as X minus x_1, \dots, x_n . We need to show there are finitely many maps from X to \mathbb{P}^1 such that the preimage of $\{0, 1, \infty\}$ is contained in $\{x_1, \dots, x_n\}$. For each degree d there are finitely many possible maps, determined by the preimages of $(0, 1, \infty)$ and their multiplicities; this is a trivial consequence of intersection theory on $X \times \mathbb{P}^1$. It remains to bound d . Now let e_i be the multiplicity with which x_i maps. We have $\sum e_i \geq 3d$ and

$\sum(e_i - 1) \leq 2d - 2 + 2g(X)$, which gives us $n \geq d + 2 - 2g(X)$, valid as long as the maps are separable, even if they are wildly ramified. \square

Now, the function x is separable (in fact tamely ramified) since φ is tamely ramified, forcing the local degree to not be divisible by p . Furthermore, $\varphi^i(x)$ is also separable, since φ is separable. We can now apply Lemma 2.7. Since the sequence of functions $\varphi^i(x)$ comes from a finite list, two functions (say $i = m, n$) have to coincide, making x persistently preperiodic after all, and the tail and cycle lengths are again fixed by the choice of m and n . Alternatively, all but finitely many functions have to be constant, and then we know the value of φ at infinitely many given points and the family C is constant, a contradiction. \square

The final step is to note that tame PCF over function fields have multipliers lying in $\overline{\mathbb{F}_p}$; this is Corollary 1.7 of [1]. Since by assumption this is not the case for C' , we have a contradiction, and Theorem 1.11 is proved.

3 Proof of Lemma 2.3

To complete the proof of Theorem 1.11, it remains to show that we can construct an infinite labeled set z_i as in Corollary 2.4 and the proof of Lemma 2.5. Trivially, for each C'' obtained from C' by marking additional repelling periodic points, we can always pick a neighborhood of each $x \in C'$ on which the map is locally invertible, since the projection map from C'' to C' is unramified. However, the proof of Lemma 2.5 works only if there exists a single neighborhood $D \ni x$ on which for each C'' the projection map is locally invertible.

In the complex case, finding such a neighborhood is trivial. If C' is a Riemann surface, then it has a universal cover \mathcal{U} with a canonical projection onto C' ; for each $x \in C'$ there is a neighborhood such that the canonical projection is locally invertible, and then we can compose with the canonical projection from \mathcal{U} to any unramified cover of C' . But in the non-archimedean case, there is no universal cover: the universal cover of Berkovich spaces exists [7, 4], but we do not know that the map $C'' \rightarrow C'$ is unramified on the level of Berkovich spaces, only that it is unramified as a map of schemes. Although invertibility is really a condition on Berkovich ramification, we will not use Berkovich language in the proof (therefore, all points are closed points of schemes) and we stay throughout in the category of finite maps (indeed, algebraic maps between affine algebraic curves, and analytic maps on affinoids).

Proof of Lemma 2.3. We will show that if D is a \mathbb{P}^1 -disk in C' then its preimage disks map to it with local degree 1. To do that, we use a variant of infinite descent. Suppose on the contrary that the local degree is more than 1. We let D be a minimal disk with this property, and show first that if the local degree is not divisible by p then there have to be critical points in the preimage of D . Then if the local degree is divisible by p , we show that we can construct a map from the preimage D' of D to \mathbb{P}^1 defined by the periodic point that we mark,

and associate a preimage of a critical point to each periodic point in such a way that if the map $\pi : D' \rightarrow D$ isn't tame, the underlying maps φ are not tame.

Now we obtain an analytic function π from the disk D' to the disk D . Imitating Faber's proof for rational functions [9], we have the following sublemma:

Lemma 3.1. *Let π be an analytic function from a \mathbb{P}^1 -disk D' onto a \mathbb{P}^1 -disk D . If the local degree is more than 1, and π is tame, then π has a critical point in D' .*

Proof. See Lemma 6.3.2 of [2] in the case of Galois covers.

In the non-Galois case, we use Newton polygons. Recall that the Newton polygon of a power series $\pi(z) = \sum a_i z^i$ is the lower convex hull in \mathbb{R}^2 of the set $\{(i, v(a_i)) : i \in \mathbb{N}\}$ where v is the valuation, i.e. $v(x) = -\log_p |x|$. A segment of the Newton polygon is maximal straight line segment; a pivot is the boundary between two segments or an endpoint of the polygon; each segment has a length, by which we mean horizontal length, and a slope, which could be negative infinite. The Newton polygon has the following relationship with the roots of the power series π :

1. If the Newton polygon of π has a segment of slope r and finite length m , then there are exactly m roots of π of valuation $-r$.
2. If the Newton polygon of π has a segment of slope r and infinite length, then it has roots of valuation $-r$ if and only if the power series converges when $v(x) = -r$. In that case the number of roots is the horizontal distance between the pivot at the left end of the segment and the rightmost point $(j, v(a_j))$ that is on rather than strictly above the segment.

Remark 3.2. A power series converges for x with valuation $-r$ if and only if the following condition is true: for every line l in \mathbb{R}^2 of slope r , there are finitely many points of the Newton polygon lying on or below l . In particular, if there is an infinite segment of slope r and the power series converges, then there are finitely many points lying on the line segment, and it makes sense to talk about the rightmost point.

The above facts can be readily verified by conjugating and multiplying by a constant so that the line segment in question has slope 0 and lies on the x -axis.

Now, we conjugate to assume $D' = D(0, 1)$ and $\pi(0) = 0$. If the local degree at 0 is more than 1 then there is nothing to prove, so assume it is 1, so that $(1, a_1)$ is a pivot. If π is tame, then whenever $(i, v(a_i))$ is a pivot, $p \nmid i$, or else the local degree on the closed disk around 0 radius equal to the absolute value corresponding to the segment left of i is divisible by p . This means that $\pi'(z) = \sum i a_i z^{i-1}$ has the same pivots as $\pi(z)$ shifted one unit to the left, and so it also has the same Newton polygon shifted one unit to the left.

If the Newton polygon of π has a finite segment of slope r for any $-\infty < r \leq 0$, then the Newton polygon of π' also has a finite segment of slope r , and thus there are critical points in D' . If the Newton polygon of π has an infinite

segment of slope 0 (we cannot have $r < 0$ or else π isn't defined on all of $D(0, 1)$) then again the same is true of the Newton polygon of π' ; if there are finitely many points $(i, v(a_i))$ on or below a horizontal line l , then there are also finitely many such points $(i - 1, v(ia_i))$ since $v(ia_i) \geq v(a_i)$, and again there are critical points in D' .

But by assumption, the local degree of π is more than 1. Therefore the Newton polygon has to have some segments of nonpositive slope apart from the length-1 segment of slope $-\infty$, and therefore π' has a root in D' . \square

Now, let D' map to D with local degree q . We will show that if $q > 1$, then D' is not a connected affinoid.

Because we obtain C'' from C' by marking periodic points, there exists a map from C'' to \mathbb{P}^1 sending (φ, x) to x , which we denote by θ . In fact we can enlarge C'' , and thus D' , to include a complete set of all points of specified formal period and multiplier; this turns it into an irreducible Galois cover. We obtain q maps on D' , $\theta_1, \dots, \theta_q$, as well as q maps on D corresponding to the symmetric functions in the θ_i s. We w

The maps θ_i all have the same image in \mathbb{P}^1 . This is because we have a map $\theta_1 \times \dots \times \theta_q : D' \rightarrow \mathbb{P}^1$, and then to show θ_i and θ_j have the same image for each i and j we compose with a Galois automorphism mapping θ_i to θ_j . Note that we can apply a Möbius transformation to fix ∞ to be a marked fixed point, which cannot collide with any of the periodic points θ_i , and therefore the maps θ_i do not map over all of \mathbb{P}^1 . Since D' is a connected affinoid this common image is a disk in \mathbb{P}^1 , which we can assume after a Möbius transformation is $D(0, 1)$.

Recall that by assumption any proper subdisk of D , including in particular any of its open disk residue classes, has preimages in D' that map with local degree 1. Pick one such open disk, call it E ; note that we have a map from E to each of its preimage disks under π , and this allows us to extend the maps θ_i to E , where they will range over open disk residue classes of $D(0, 1)$.

Now, because we marked the critical points in C' , we have an analytic map from D to each critical point, as well as an analytic map from D' to each critical point. Let i be the minimal number such that for some $x \in D'$ the disk $\varphi^i(D(0, 1))$ contains a critical point γ ; since $D(0, 1)$ is a disk around points in the Julia set, i is guaranteed to exist. Because φ is tame, we can apply Lemma 3.1 and obtain that φ^i maps $D(0, 1)$ onto its image with local degree 1.

We will show that $\varphi^{-i}(\gamma)$ defines an analytic map from D' to $D(0, 1)$. But first, we assume that a map exists, marking $\varphi^{-i}(\gamma)$ if necessary. For every $x \in D'$ we have $\theta_j(x) - \varphi^{-i}(\gamma)(x) \in \mathbb{A}^1 \setminus \{0\}$ since critical points and repelling points do not collide, and the i th image of a repelling point is still a repelling point. Hence the map $\theta_j - \varphi^{-i}(\gamma)$ maps over some disk $D(a, r)$ with $r < |a|$, and since $D(a, r)$ intersects $D(0, 1)$, we have $D(a, r) \subsetneq D(0, 1)$. But now the map $\varphi^{-i}(\gamma)$ can be constructed from analytic maps and from maps that are inverse to invertible (i.e. local-degree-1) maps, so it is analytic.

We can now conjugate each $x \in D'$ by a Möbius transformation by $\varphi^{-i}(\gamma)$ to place $\varphi^{-i}(\gamma)$ persistently at 0, so that each θ_j maps D' into $D(b_1, s_1)$ with $s_1 < |b_1| \leq 1$. By assumption, all θ_j s map D' into the same disk $D(b_1, s_1)$. Since

the conjugation $z \mapsto z - \varphi^{-i}(\gamma)$ preserves distances, this means that we have shown that $\max\{|\theta_i(x) - \theta_j(x)| : x \in D', i \neq j\} \leq s_1 < 1$. Observe that we can repeat this process of conjugation any finite number of times, such that each θ_j maps D' into $D(b_l, s_l)$ with $s_l < |b_l| \leq s_{l-1}$. Note that the earlier preimages of critical points are moved away from zero, but are still not contained in $D(b_l, s_l)$.

If we can force the radius s_l to be too small, then we will obtain a contradiction. Although we cannot force s_l to approach zero, we will use the fact that, by the minimality of i in the construction of $\varphi^{-i}(\gamma)$, the order of the backward image i is nondecreasing in l , and cannot stabilize since there are only finitely many critical points but must instead go to infinity. In particular, any point in $D(0, 1)$ in the backward image of a critical point will be reached for some l . We will construct a point in the backward orbit of a critical point that is closer to θ_1 than any other θ_j .

Namely, let r_i now be $\min\{|\theta_j(x) - \theta_i(x)| : x \in D', i \neq j\}$. Because C'' is an irreducible Galois cover of C' and D' is a connected affinoid, the quantity r_i is symmetric in i and thus independent of i , and we denote it by r . Let $x \in D'$, and consider the minimal i such that $\varphi^i(D^-(\theta_1(x), r))$ contains a critical point γ , which is again guaranteed to be a finite integer. After finitely many Möbius transformations placing lower-order preimages of critical points away from the repelling points, $\varphi^{-i}(\gamma)$ will be an analytic map on D' . It also satisfies $|\varphi^{-i}(\gamma)(x) - \theta_1(x)| < r \leq |\theta_j(x) - \theta_1(x)|$ for all $x \in D'$ and $j \neq 1$. Now after a final Möbius transformation placing $\varphi^{-i}(\gamma)$ at zero for all $x \in D$, we get that θ_1 maps over a subdisk of $D^-(0, r)$ while every other θ_j maps over a disk containing a point of radius at least r , contradicting the assumption that all θ_j s have the same image. \square

References

- [1] Robert Benedetto, Patrick Ingram, Rafe Jones, and Alon Levy, *Critical orbits and attracting cycles in p-adic dynamics*, arXiv:1201.1605, Sep 2012.
- [2] Vladimir Berkovich, *Etale cohomology for non-archimedean analytic spaces*, IHES Publ. Math.
- [3] ———, *Spectral theory and analytic geometry over non-archimedean fields*, Mathematical Surveys and Monographs, vol. 33, American Mathematical Society, 1990.
- [4] ———, *Smooth p-adic analytic spaces are locally contractible*, Invent. Math. **137** (1999), no. 1, 1–84.
- [5] Jean-Paul Bézivin, *Sur les points périodiques des applications rationnelles en dynamique ultramétrique*, Acta Arith. **100** (2001), no. 1, 63–74.
- [6] Eva Brezin, Rosemary Byrne, Joshua Levy, Kevin Pilgrim, and Kelly Plummer, *A census of rational maps*, Conformal Geometry and Dynamics **4** (2000), 35–74.

- [7] A. J. De Jong, *Etale fundamental groups of non-archimedean analytic spaces*, Comp. Math. **97** (1995), 89–118.
- [8] Adrien Douady and John H. Hubbard, *A proof of Thurston’s topological characterization of rational functions*, Acta Math. **171** (1993), 263–297.
- [9] Xander Faber, *Topology and geometry of the berkovich ramification locus for rational functions, ii*, Math. Ann. (2012), 1–26.
- [10] Benjamin Hutz and Michael Tepper, *Multiplier spectra and the moduli space of degree 3 morphisms on \mathbb{P}^1* , arXiv:1110.5082, Oct 2011.
- [11] Curtis T. McMullen, *Families of rational maps and iterative root-finding algorithms*, Ann. of Math. (2) **125** (1987), no. 3, 467–493. MR MR890160 (88i:58082)
- [12] Joseph H. Silverman, *The space of rational maps on \mathbf{P}^1* , Duke Math. J. **94** (1998), no. 1, 41–77. MR MR1635900 (2000m:14010)

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